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Logics and Quantum Gravity

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We consider the logic needed for models of quantum gravity, taking as our starting point a simple pregeometric toy model based on graph theory. First a discussion of quantum logic seen in the light of canonical quantum gravity is given, then a simple toy model is proposed and the logical structure underlying it exposed. It is then shown that this logic is nonclassical and in fact contains quantum logics as special cases. We then go on to show how Yang–Mills theory and quantum mechanics fits in. A single mathematical structure is proposed capable of containing all these subjects in a natural and elegant way. Causality plays an important role. The mere presence of a causal relation almost inevitably yields this kind of logic.

1. INTRODUCTION AND SETUP

It has been known for several years that quantum theory can be formulated in a beautiful manner using nonclassical logics (Gudder, 1979; Varadarajan, 1985; Pitowsky, 1989). Now, to me the greatest unsolved problem in quantum theory is the quantization of gravity. It has long been believed that in order to perform this quantization we have to give up our usual notions of space and time, and that this breaking up of concepts occurs at the Planck scale. Most models claim that at this scale space and time should become discrete, in other words the continuum of general relativity is replaced, at the Planck scale, by a discrete structure.

Canonical quantization of Einstein's theory of gravitation leads to the Wheeler–DeWitt equation, which has the form of a time-independent Schrödinger equation, but with the metric tensor g_{ij} as parameter, i.e.,

$$\mathscr{H}\Psi = (-\Delta + \sqrt{g}R^{(3)})\Psi = 0 \tag{1}$$

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where Δ is some second-order differential operator

$$\Delta = \frac{1}{2} \frac{1}{\sqrt{g}} G_{ij;kl} \frac{\delta^2}{\delta g_{ij} \delta g_{kl}} \tag{2}$$

where g_{ij} is the three-metric and $G_{ij;kl}$ is the so-called *supermetric.* This is the basic equation of quantum cosmology and Hawking's Euclidean quantum gravity program. An essential feature is the lack of a time derivative; time does not exist at this level. An important conceptual problem arises: the Wheeler–DeWitt equation describes the entire universe *including* the observer—we have no Heisenberg cut, between observer and observed. Quantum logic is inspired by old-fashioned quantum theory with its Heisenberg cut, and it thus becomes necessary to study which alterations must be made.

2. ON QUANTUM LOGIC

Quantum logic is by now a well-established area of research, with its own standardized notions. The reason for giving a short review of it anyway is in order to clarify its connection to the Copenhagen interpretation and hence the so-called Heisenberg cut, which separates observer and observed (to some extent: the observer interacts with what is observed). A second and very important reason is the consideration of *alternative* logics; while it is clear that quantum logic must somehow violate some of the Boolean axioms, it is not clear which ones. Normally the violated axiom is taken to be the law of distributivity, but others could just as well have been chosen, and we will make some comments on this. Since we want to study quantum gravity, it is also important to consider the connection between logic and topology/geometry. This can be done by defining *semantics*, and we will see that various topological/algebraic structures have associated to them various kinds of logic. Finally we will make some comments on which kinds of logic are equivalent in some way or other, and which is the more general. The basic references for this section are Gudder (1979), Varadarajan (1985), and Pitowsky (1989).

Sometimes Niels Bohr referred to quantum theory as a theory telling us what we could get out of observations, more than a theory telling us what is "really" going on: the experimental setup was assumed to follow classical laws. According to many formulations, the interaction of the quantum system with this forced the wave function to collapse into an eigenstate of the appropriate operator; how this collapse was to take place was beyond the power of quantum theory to describe. This has led a number of researchers to search for alternative formulations or even alternative theories (hidden variables, many-world interpretation, etc.).

And clearly this formulation of quantum theory is inappropriate for a study of gravity, where we have to take the entire universe into account, and are thus not allowed to make the Heisenbergian distinction between an observer governed by the laws of classical physics and an observed object governed by the laws of quantum theory.

The starting point of quantum logic is an analysis of the observation process. For an analysis of this we need a set Q of allowed physical questions. Let us first consider what Q is. Any physical question can be written in the form, "does the physical parameter ξ take its value in the region Δ?," i.e., Δ represents the uncertainty of the measurement of ξ. Hence the elements of Q must be pairs (ξ, Δ). The map m then performs the measurement by assigning probabilistic values to the statement $\xi \in \Delta$. This gives us a natural partial ordering:

$$(\xi_1, \Delta_1) \leq (\xi_2, \Delta_2) \Leftrightarrow \xi_1 = \xi_2 \wedge \Delta_1 \subseteq \Delta_2 \tag{3}$$

and we have a natural lattice structure as follows:

$$(\xi, \Delta_1) \wedge (\xi, \Delta_2) \equiv (\xi, \Delta_1 \cap \Delta_2) \tag{4}$$

$$(\xi, \Delta_1) \vee (\xi, \Delta_2) \equiv (\xi, \Delta_1 \cup \Delta_2) \tag{5}$$

Since any physical value has to be a real number, we can, without loss of generality, assume $\Delta \subseteq \mathsf{R}$. This gives us a *complemented lattice* with

$$(\xi, \Delta)' \equiv (\xi, \mathsf{R}/\Delta) \tag{6}$$

For technical reasons one requires Δ to be a Borel set.

A quantum logic is defined as a lattice with a zero element 0 ($\forall a\colon a \leq 0 \Rightarrow a = 0$) and a unit element 1 ($\forall a\colon 1 \leq a \Rightarrow a = 1$) satisfying:

1. The lattice must be *orthocompleted*, i.e., to each element a a unique element a' exists such that $a \wedge a' = 0$ and $a \vee a' = 1$.
2. The lattice must be *orthomodular*, i.e., $a \leq b \Rightarrow b = a \vee (b \wedge a')$.
3. The lattice must be σ-*orthocomplete*, i.e., $\forall a_i \in \mathscr{L}\colon a_i \perp a_j, i \neq j \Rightarrow \bigvee a_i \in \mathscr{L}$.

Here $a \perp b$ means $a \leq b'$. The first condition is not as innocent as it looks; it amounts to demanding, for each $a \in \mathscr{L}$, the existence of another element a' such that $a \wedge a' = 0$ and $a \vee a' = 1$. This is essentially a classical requirement. We will return to this point later. The second condition is a weakening of the distributive law $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, which we do *not* impose, under the assumption that $a \wedge a' = 1$. Hence this requirement is also dubious. The third condition looks like a technical convenience but we will comment on its implications later.

We have now defined what we mean by a universe Q of physical questions; we must make the connection to the measuring process. Consider a question $q \in Q$; an experiment then gives a probability for q to hold, i.e., if $q = (\xi, \Delta)$, then we get the probability for $\xi \in \Delta$. Given a poset $\mathscr{L}$, a map $m: \mathscr{L} \to [0, 1]$ is a *probability measure* if:

1. $m(1) = 1$.
2. $m(\bigvee a_i) = \sum m(a_i)$ if $a_i \perp a_j$, $i \neq j$.

The first requirement is just the obvious normalization [the question 1 corresponds to (ξ, R) and the requirement thus simply states that all physical parameters must take their values in the real line]. The second allows us to make consecutive measurements of noninterfering variables (we will return to this). An *event structure* is then a pair $(\mathscr{L}, \mathscr{M})$ where $\mathscr{L}$ is a quantum logic and $\mathscr{M}$ is a set of *order-determining* probability measures, i.e., $\forall m \in \mathscr{M}: m(a) \leq m(b) \Rightarrow a \leq b$. This is equivalent to (Gudder, 1979) the requirements:

1. $\forall m \in \mathscr{M}: m(a) = m(b) \Rightarrow a = b$.
2. If $a_1, a_2, \ldots \in \mathscr{L}$ satisfy $m(a_i) + m(a_j) \leq 1$ *for all* $i \neq j$ and all $m \in \mathscr{M}$, then a $b \in \mathscr{L}$ exists such that

$$m(b) + \sum_i m(a_i) = 1$$

   for all $m \in \mathscr{M}$.

The elements of $\mathscr{M}$ are known as *states*. The first requirement is very natural: we identify questions which cannot be distinguished by any state. The other requirement is somewhat more dubious. It states that given two, say, questions a_1, a_2 such that $m(a_1) + m(a_2) \leq 1$ for all states m, i.e., when a_1 occurs with certainty in a state m $[m(a_1) = 1]$, then we know with certainty that a_2 does not occur $[m(a_2) = 0]$; the requirement is then that a third experiment b always exists such that $m(a_1) + m(a_2) + m(b) = 1$ for all states. I find this requirement rather strange. It is a consequence of the σ-orthocompleteness, which can be seen by noting that $m(a_i) + m(a_j) \leq 1$ for all states m implies $m(a_i) \leq 1 - m(a_j) \equiv m(a_j')$; hence $a_i \perp a_j$ and by σ-orthocompleteness $c = \bigvee a_i$ exists and by complementarity so does $b = c'$ and we get $m(b) + \sum_i m(a_i) = m(c' \vee c) = 1$. For a finite set of a_i, the requirement follows from $a' \vee a = 1$, since this gives $1 = m(a' \vee a) = m(a) + m(a')$, as $a \perp a'$ always holds. We will have reason to comment more on this requirement later.

The prototype of a quantum logic is the set of orthogonal projections of a Hilbert space; actually, a quantum logic is a formalization of this structure, which is perhaps unfortunate, as the Hilbert space formulation is just a formulation, a derived concept, useful for computational purposes well suited to our classically conditioned way of thinking (a Hilbert space is the more or less straightforward generalization of Euclidean space). A discussion of Bell inequalities in the setting of quantum logics can be found in Pulmannová and Majernik (1992).

3. A SIMPLE MODEL

The last section gave us all the needed formalism, and we will now set up a simple toy model. This model will resemble canonical quantum gravity, but will be discrete; in fact it will be a pregeometric model. We will then consider the underlying logical structure of this, and we will see how this differs from ordinary quantum logic.

As fundamental objects we take *points* and *links* between these, and we define a set of operators $a, a^\dagger, b, b^\dagger$ to annihilate and create these objects. The universe is thus represented by a *graph*. Only the spatial part is thus represented—this is a kind of discretized version of Wheeler–DeWitt quantum gravity. At each time step one of the possible operations is chosen at random subject to the following rules:

1. Between any two points there can be at most one link.
2. No link can have the same point as its respective ends.
3. It is forbidden to attempt to annihilate a link or a point when no such object exists.
4. Similarly, it is forbidden to attempt to create a link when all possible links exist (i.e., when the graph is complete, a simplex).
5. When deleting a point, one of those with the lowest *degree* (i.e., the lowest number of links emanating from it) is chosen at random.

The set of all graphs is denoted by Γ and is referred to as *metaspace*, to show the similarity with Wheeler's superspace. Note that no restraints on the dimensionality of the graphs are imposed; in fact, the dimension is allowed to vary, not only from region to region, but also from time to time. This model has been proposed and studied in Antonsen (1992, 1994, n.d.) and I refer to these for its predictions of such quantities as dimension and Euler–Poincaré characteristic and for its connections with other models in the literature.

Let us at once note that we have a natural partial ordering, namely $\subseteq$. Notice also that we have a natural null element $\varnothing$, the empty graph, but we have no maximal element, since given any graph G, we can always construct new graphs which are bigger, e.g., $G \cup G$ or $G \cup H$, H any graph. Now, for reasons of causality only regions connected to each other can possibly communicate, i.e., we cannot "see" a region of space which is not connected to our own. This suggests the introduction of an equivalence relation. Define

$$G \sim H \Leftrightarrow \exists n: \quad G = H \cup \{p_1, \ldots, p_n\} \vee H = G \cup \{p_1, \ldots, p_n\} \qquad (7)$$

where G, H are arbitrary graphs and $p_1, \ldots, p_n$ are isolated points and where the unions are understood as being disjoint. Thus we have equivalence classes

$$[G] = G \cup \mathscr{P}(\mathsf{N}) \qquad (8)$$

In graph theory there is a natural concept of duality, namely the *complement* of a graph. G'' is defined as that graph which has the same set of points as G but in which two points are linked if and only if they are *not* linked in G. Hence $G \cap G'$ is the set of points, and since these are isolated, we have

$$[G] \cap [G'] = [\varnothing] \qquad (9)$$

Letting $|G|$ denote the *order* of a graph, i.e., the number of points in it, and letting Δ_n denote the n-simplex, i.e., the graph with n points in which all pairs of points are linked, we have

$$[G] \cup [G'] = [\Delta_{|G|}] \qquad (10)$$

We see that we have some problems with this. While $x \wedge x' = 0$ is certainly one of the demands we impose on a (logical) complement, the other, $x \vee x' = 1$, is certainly not fulfilled. Also note that $x'' \neq x$. But what is worse, we have not really defined the equivalence class 1. We can make another attempt at defining a complement.

Write $\bar{\Gamma} = \Gamma/\sim$. Define the *relative pseudocomplement* (Goldblatt, 1984; Bell, 1988; Chapman and Rowbotton, 1992; Vickers, 1989) of two graphs G, H as

$$[G]_{[H]} = \max\{[K] \in \bar{\Gamma} | [G] \cap [K] \leq [H]\} \qquad (11)$$

where $\leq$ is the partial ordering inherited from Γ. Defining

$$\neg[G] = [G]_{[\varnothing]} \qquad (12)$$

we have the essential property

$$\neg\neg[G] \neq [G] \qquad (13)$$

so considering $\neg$ as a complement, our algebra is not Boolean. In the next section we investigate non-Boolean logics to find out what kind of logics we *do* have, and we will also see how to define a unit 1.

4. NONCLASSICAL LOGICS: INTUITIONISM AND MODALITIES

Classical logic is described by Boolean lattices, i.e., by lattices B which are:

1. Complemented ($\neg a$ is defined for all a).
2. Distributive [for all a, b, c the law of distributivity, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, holds].

Clearly, we can generalize in three ways: (1) discard the first axiom, (2) discard the second, or (3) throw both away. The last just gives us a general lattice and is not of any particular interest; a lattice is too weak a structure to yield interesting results. As we say earlier, quantum logic chooses the second alternative [in its standard formulation, other versions have been proposed (Garden, 1984)]. Here we will now pursue the first.

A distributive lattice is known as a *Brouwerian algebra*. Now, a complement satisfies per definition $a \wedge \neg a = 0$ and $a \vee \neg a = 1$; the second requirement is also known as "the law of excluded middle" or "*tertium non datur*" [=a third (possibility) is not given]; it is essential to classical thought. Let us throw away this last requirement, and instead define a *relative pseudocomplement* $a'_{(b)}$ as

$$a'_{(b)} \equiv \sup\{x \mid a \wedge x \leq b\} \tag{14}$$

where $\leq$ is the partial ordering of the lattice. The natural interpretation of this quantity is the implication $a \Rightarrow b$. Define also the *pseudocomplement* $\neg a$ as $a'_{(0)}$; in a Boolean algebra this would become the usual negation. In any relative pseudocomplemented lattice L the following hold $\forall a, b, c \in L$:

1. $(a \Rightarrow a) = 1$.
2. If $a \leq b$, then $(a \Rightarrow b) = 1$.
3. $b \leq (a \Rightarrow b)$.
4. $(a \wedge (a \Rightarrow b)) = a \wedge b \leq b$.
5. $(a \Rightarrow b) \wedge b = b$.
6. $(a \Rightarrow b) \wedge (a \Rightarrow c) = (a \Rightarrow (b \wedge c))$.
7. $(a \Rightarrow b) \leq ((a \wedge c) \Rightarrow (b \wedge c))$.
8. If $b \leq c$, then $(a \Rightarrow b) \leq (a \Rightarrow c)$.
9. Semitransitivity: $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq (a \Rightarrow c)$.
10. $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq ((a \vee b) \Rightarrow c)$.
11. $a \Rightarrow (b \Rightarrow c) \leq (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$.

A relative pseudocomplemented lattice with a zero is known as a *Heyting* *algebra* after Brouwer's main pupil. One can prove

$$\neg 0 = 1 \quad \text{and} \quad \neg 1 = 0 \tag{15}$$

$$\textit{if} \quad \neg a = 1 \quad \text{then} \quad a = 0 \tag{16}$$

$$a \leq \neg\neg a \quad \text{and} \quad \neg a = \neg\neg\neg a \tag{17}$$

$$(a \Rightarrow b) \leq (\neg b \Rightarrow \neg a) \tag{18}$$

$$a \wedge \neg a = 0 \quad \text{but} \quad (\neg\neg(a \vee \neg a) = 1 \tag{19}$$

$$\neg(a \vee b) = \neg a \wedge \neg b \tag{20}$$

$$\neg a \vee \neg b \leq \neg(a \wedge b) \tag{21}$$

$$\neg a \vee b \leq a \Rightarrow b \tag{22}$$

$$\neg a \leq (a \Rightarrow b) \tag{23}$$

$$(a \Rightarrow b) \wedge (a \Rightarrow \neg b) = \neg a \tag{24}$$

Notice the weakened version of the law of excluded middle above, as well as the weakening of the classical requirement $a = \neg\neg a$ and of De Morgan's law $\neg(a \vee b) = \neg a \wedge \neg b$. Any relative pseudocomplemented lattice has a unit, namely

$$1 \equiv (a \Rightarrow a) = a'_{(a)} \tag{25}$$

for any a. Also, if $\forall a$: $\neg\neg a = a$, then the algebra is Boolean. Note, furthermore, the interesting fact that $\neg$ maps the Heyting algebra into a Boolean subalgebra: $\neg\neg(\neg a) = (\neg a)$.

It should be stressed that in a general Heyting algebra the list of logical connectives $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\neg$, $\forall$, $\exists$ is *not* redundant as in the case of classical logic—and quantum logic!—where, given $\wedge$, $\neg$, say, we can construct $\vee$, $\Leftrightarrow$, $\Rightarrow$. For instance,

$$(b \vee c) \equiv (\neg((\neg b) \wedge (\neg c)))$$

$$(b \Rightarrow c) \equiv ((\neg b) \vee c)$$

$$(b \Leftrightarrow c) \equiv ((b \Rightarrow c) \wedge (c \Rightarrow b)$$

Other theorems which hold in classical as well as quantum logic are

$$a \vee \neg a = 1$$

$$a \Rightarrow (a \vee b)$$

$$a \wedge b \Rightarrow a$$

$$\neg\neg a = a$$

$$\neg(a \wedge b) = (\neg a) \vee (\neg b)$$

$$((a \wedge b) \vee (a \wedge c) \Rightarrow a \wedge (b \vee c)$$

Only the second and third hold in an arbitrary Heyting algebra.

Example 1. An example which also elucidates the contents of Heyting algebras is the unit interval $I = [0, 1]$ with the lattice structure given by

$$x \wedge y \equiv \min\{x, y\}$$
$$x \vee y \equiv \max\{x, y\}$$

The relative pseudocomplement is well defined for all pairs x, y and is in fact

$$(x \Rightarrow y) = x'_{(y)} = \begin{cases} 1 & x \leq y \\ y & \text{otherwise} \end{cases}$$

It is easily checked that this is in fact a Heyting algebra.

Example 2. Consider the category of modules over some ring R. The dual of a module M is defined as $M^* = \mathrm{Hom}(M, R)$ and in general $M \subseteq M^{**}$. (A similar situation occurs in the theory of Banach spaces, where $X \subseteq X^{**}$ holds in general and equality only holds for a special class, namely the *reflexive spaces*, e.g., for Hilbert spaces). If we restrict ourselves to modules with rank at most n, where n is some fixed number, then we get a lattice. $M \cap M^*$ is the zero-module.

Example 3. A topological space (X, τ) is also an example of a Heyting algebra. Let the pseudocomplement be given by $\neg A = (X \backslash A)^\circ$, where A° denotes the interior of the open set $A \in \tau$; then $\neg\neg A = \bar{A}^\circ$, so $A \subseteq \neg\neg A$, and the algebra of open sets become a Heyting algebra. The sets A for which equality holds are known as regular open sets, so the algebra is Boolean if and only if all open sets are regular.

Example 4. The Logic of Finite Information. Consider a setup consisting of an apparatus which measures some system. And suppose this measurement yields a sequence x_n of zeros and ones, i.e., each measurement gives a simple "yes," "no" answer (the actual number of possible values for x_n is immaterial: 0, 1 is mere convenience). The entire measurement can then be represented by a bit stream, and "$x_n = 0$" then means that the nth bit was read as being zero. Consistency demands that $(x_n = 0) \wedge (x_n = 1) = \bot$, where $\bot$ denotes a false statement (similarly $\top$ denotes a true statement, $\top = \mathit{true}$, $\bot = \mathit{false}$). Demanding that we cannot read a bit before having read the one just before it (we cannot jump ahead in "time," i.e., if we have a knowledge of the system at "time" n, then we must also have some knowledge of the system at "time" $n - 1$), we cannot have the law of excluded middle $(x_n = 0) \vee (x_n = 1) = \top$ since the statement $(x_n = 0) \vee (x_n = 1)$ only means that we have read the nth bit (otherwise the statement would be meaningless), but that we do not state the result of the reading. This implies

$$[(x_{n+1} = 0) \vee (x_{n+1} = 1)] \leq [(x_n = 0) \vee (x_n = 1)] \tag{26}$$

One can prove that this actually gives a Heyting algebra [see Vickers (1989), from which this example is essentially taken]. This example is interesting because a recent article by Doebner and Lücke (1991) shows that quantum logic can be formulated as resulting from classical logic by imposing some concept of realistic measurement, essentially equivalent to finiteness of the amount of observations which can be made.

Example 5. Consider a poset P; as is well known, the so-called *Alexandrov sets*, $A^+(a) = \{b \in P | a \preceq b\}$, form a topology. These sets also form a Heyting algebra, and we thus see a connection between causality (a partial order), topology (the corresponding Alexandrov sets), and Heyting algebras, which suggests that Heyting algebras are somehow related to causality. Further hints of such a relationship will be found later.

While classical logic, CL, is connected with Boolean algebras, the kind of logic related to Heyting algebras is known as Intuitionistic logic, IL. It has been proven that, e.g., CL-number theory is contained within IL-number theory (K. Gödel), and it has been suggested that *the entire system of CL is contained in a special sector of IL.* Intuitionistic logic is thus very general and very powerful (Kleene, 1980; see also Goldblatt, 1984; Dummet, 1977; Troelstra and van Dalen, 1988).

We get a clue about its interpretation from Example 1 above. The unit interval I is a Heyting algebra, and can thus be used in the assigning of truth values; this suggests a multivalued logic, and perhaps a connection with probability.

4.1. Formal Languages

By a (*formal*) *language* $\mathscr{L}$ we mean a set with:

1. An *alphabet* $\mathscr{A}, \mathscr{B}, \ldots$ of which valid statements are built.
2. A set of rules specifying how to formulate well-formed sentences; a *syntax*.
3. A set of such well-formed statements, the *axioms*.
4. A set of rules specifying how to construct new well-formed statements from old ones, the *transformation rules*.

The usual transformation rules are:

1. $\neg\mathscr{A}$ is well formed when $\mathscr{A}$ is.
2. $(\mathscr{A} \vee \mathscr{B})$, $(\mathscr{A} \wedge \mathscr{B})$, and $(\mathscr{A} \Rightarrow \mathscr{B})$ are well formed whenever $\mathscr{A}, \mathscr{B}$ are.
3. $\forall a \colon \mathscr{A}(a)$ and $\exists a \colon \mathscr{A}(a)$ are well formed whenever $\mathscr{A}$ is a well-formed statement with a parameter a.

And the axioms of *classical logic* are of the pattern:

1. $(\mathscr{A} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{A}))$.
2. $((\mathscr{A} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})) \Rightarrow ((\mathscr{A} \Rightarrow \mathscr{B}) \Rightarrow (\mathscr{A} \Rightarrow \mathscr{C}))$.
3. $(((\neg\mathscr{A}) \Rightarrow (\neg\mathscr{B})) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{A})))$.

Let us give some examples.

Example 6. Classical proposition calculus, the Gentzer schemes:

1a. $\mathscr{A} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{A})$.
1b. $(\mathscr{A} \Rightarrow \mathscr{B}) \Rightarrow ((\mathscr{A} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})) \Rightarrow (\mathscr{A} \Rightarrow \mathscr{C}))$.
2. if $\mathscr{A}$ and $(\mathscr{A} \Rightarrow \mathscr{B})$, then $\mathscr{B}$.
3. $\mathscr{A} \Rightarrow (\mathscr{B} \Rightarrow (\mathscr{A} \wedge \mathscr{B}))$.
4a. $(\mathscr{A} \wedge \mathscr{B}) \Rightarrow \mathscr{A}$.
4b. $(\mathscr{A} \wedge \mathscr{B}) \Rightarrow \mathscr{B}$.
5a. $\mathscr{A} \Rightarrow (\mathscr{A} \vee \mathscr{B})$.
5b. $\mathscr{B} \Rightarrow (\mathscr{A} \vee \mathscr{B})$.
6. $(\mathscr{A} \Rightarrow \mathscr{C}) \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\mathscr{A} \vee \mathscr{B}) \Rightarrow \mathscr{C}))$.
7. $(\mathscr{A} \Rightarrow \mathscr{B}) \Rightarrow ((\mathscr{A} \Rightarrow \neg\mathscr{B}) \Rightarrow \neg\mathscr{A})$.
8. $\neg\neg\mathscr{A} \Rightarrow \mathscr{A}$.

Compare these with the theorems which we know hold for quantum logics.

Example 7. Some theorems from intuitionistic proposition calculus:

1. $(\mathscr{A} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})) \wedge (\neg\neg\mathscr{A}) \wedge (\neg\neg\mathscr{B}) \Rightarrow (\neg\neg\mathscr{C})$.
2. $(\neg\neg(\mathscr{A} \Rightarrow \mathscr{B})) \Rightarrow (\neg\neg\mathscr{A} \Rightarrow \neg\neg\mathscr{B})$.
3. $(\neg\neg(\mathscr{A} \Rightarrow \mathscr{B})) \wedge (\neg\neg(\mathscr{B} \Rightarrow \mathscr{C})) \Rightarrow (\neg\neg(\mathscr{A} \Rightarrow \mathscr{C}))$.
4. $\neg\neg(\mathscr{A} \wedge \mathscr{B}) \Leftrightarrow \neg\neg\mathscr{A} \wedge \neg\neg\mathscr{B}$.
5. $\mathscr{A} \Rightarrow \neg\neg\mathscr{A}$.
6. $\neg\neg\neg\mathscr{A} \Leftrightarrow \mathscr{A}$.
7. $\mathscr{A} \vee \neg\mathscr{A} \Rightarrow (\neg\neg\mathscr{A} \Leftrightarrow \mathscr{A})$.
8. $\neg(\mathscr{A} \Leftrightarrow \neg\mathscr{A})$.
9. $\neg\neg(\mathscr{A} \vee \neg\mathscr{A})$.

Uncertainties can be incorporated into a given language system by going to its *modal extension*; this consists in the introduction of two new logical operators $\Box$, $\Diamond$, with the interpretation that $\Box\mathscr{A}$ is true whenever $\mathscr{A}$ holds in any valuation of the logical system; we call this operator *necessity*. The other, *possibility*, is defined from the first as

$$\Diamond\mathscr{A} \equiv \neg(\Box\neg\mathscr{A}) \tag{27}$$

In such a modal logic, the law of excluded middle no longer holds (Goldblatt, 1984; Bell, 1988; Chapman and Rowbotton, 1992; Vickers, 1989; Kleene, 1980; Dummet, 1977; Troelstra and van Dalen, 1988). How these new operations affect the old ones has to be specified by extra axioms. The most important of these are:

$$\text{if} \quad \Box(\mathscr{A} \Rightarrow \mathscr{B}) \quad \text{then} \quad (\Box\mathscr{A} \Rightarrow \Box\mathscr{B}) \qquad \text{(K)}$$

$$\text{if} \quad \Box\mathscr{A} \quad \text{then} \quad \mathscr{A} \qquad \text{(T)}$$

$$\text{if} \quad \Box\mathscr{A} \quad \text{then} \quad \Box\Box\mathscr{A} \qquad \text{(S4)}$$

$$\text{if} \quad \mathscr{A} \quad \text{then} \quad \Box\Diamond\mathscr{A} \qquad \text{(B)}$$

$$\text{if} \quad \Diamond\mathscr{A} \quad \text{then} \quad \Box\Diamond\mathscr{A} \qquad \text{(S5)}$$

where $\mathscr{A}$, $\mathscr{B}$ are propositions, and where the letters in parentheses are the names of the axioms. The list is in order of decreasing generality.

Just as we saw that Heyting algebras are related to topologies, modality is closely connected to a topological concept, namely *closure spaces*. Given a modality $\Diamond$, we can construct the sets

$$\bar{X} = \{x \mid \Diamond(x \in X)\} \tag{28}$$

consisting of the elements which possibly belong to X (a kind of *fuzzy set*, which we will return to later). One can now prove

$$X \subseteq \bar{X} \tag{29}$$

$$X = \bar{X} \tag{30}$$

$$\overline{X \cap Y} = \bar{X} \cap \bar{Y} \tag{31}$$

$$X \subseteq Y \Rightarrow \bar{X} \subseteq \bar{Y} \tag{32}$$

which are just the McKinsey–Tarski axioms for a closure space. There is indeed a 1–1 correspondence between closure spaces and modalities.

The class of modalities on a logic forms a lattice in themselves, in which the zero element is the modality *fix*, defined by $fix(\omega) = \omega$, and the unit is the modality *true*: $\omega \mapsto \mathsf{T}$, where T denotes a true statement. Given sentences α, β, we can construct modalities

$$\mu^{\alpha}(\beta) = \alpha \vee \beta \tag{33}$$

$$\mu_{\alpha}(\beta) = (\alpha \Rightarrow \beta) \tag{34}$$

$$\mu_{\neg\neg}(\alpha) = \neg\neg\alpha \tag{35}$$

One can prove that a theory is classical if and only if, for all modalities μ, sentences α, β exist such that $\mu_{\alpha} = \mu^{\beta}$, i.e., provided $\exists\alpha, \beta\forall\omega: (a \Rightarrow \omega) = (\beta \vee \omega)$. To see that this holds when the logic is classical is trivial: just choose $\beta = \neg\alpha$; the other way is much more complicated and I refer to the literature (Goldblatt, 1984; Bell, 1988; Chapman and Rowbotton, 1992; Vickers, 1989).

The interpretation of these kinds of logic is the subject of the next section.

4.2. Semantics: Kripke and Beth

For Boolean logics we map the lattice of propositions into some Boolean lattice in such a way that true statements are mapped into 1 and false ones into 0. Such a map is called a *valuation* and we write

$$B \models_v \mathscr{A} \tag{36}$$

if the proposition $\mathscr{A}$ gets mapped by the valuation v into the unit element of the Boolean algebra B. If any valuation v gives $v(\mathscr{A}) = 1$, then we just write $B \models \mathscr{A}$, and finally if this holds for any Boolean algebra whatsoever, we write $\models \mathscr{A}$, and $\mathscr{A}$ is called a *tautology*. When working with different kinds of logic we sometimes want to specify that $\mathscr{A}$ is a tautology when using any *Boolean algebra*, and in this case we write $BA \models \mathscr{A}$.

As Gödel has proven, we have to distinguish between *provable* statements and *true* ones, as not all true statements are necessarily provable (the famous Gödel theorem). Let us be more formal.

A proof is a *finite* sequence of well-formed statements which are either (i) axioms or (ii) derived from axioms by the transformation rules defined above. If a well-formed statement $\mathscr{B}$ follows from the sequence $\mathscr{A}_1, \ldots, \mathscr{A}_n$ then we write

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \vdash \mathscr{B} \tag{37}$$

Note that the axioms can be characterized by $\vdash \mathscr{A}$. When we want to specify the language we write $\vdash_{\mathscr{L}}$ instead of just $\vdash$. Clearly we must demand that

$$\vdash \mathscr{B} \quad \text{then} \quad \models \mathscr{B} \tag{38}$$

We say that a theory is *sound* if the reverse also holds, i.e., if a statement is true if and only if it can be proven. The symbol $\vdash$ is read "yields" and $\models$ is read "models."

How are we to generalize these metamathematical notions to modal and other nonclassical logics? When the logic is described not by Boolean algebras but by some other kind of lattice structure, then the generalization is obvious. For instance, a valuation in a Heyting algebra is a map $n: \mathscr{L} \to H$, where H is some Heyting algebra; such maps are known as *H-valuations*, and we write $H \models_v \mathscr{A}$ if and only if $v(\mathscr{A}) = 1 \in H$; if it holds for any H-valuation we simply omit the v-index; and if it holds in any

Heying algebra we write $HA \vDash \mathscr{A}$. It can be proven that

$$BA \vDash \mathscr{A} \qquad \text{if and only if} \quad \vdash_{\text{CL}} \mathscr{A} \tag{39}$$

$$HA \vDash \mathscr{A} \qquad \text{if and only if} \quad \vdash_{\text{IL}} \mathscr{A} \tag{40}$$

where $\vdash_{\text{CL}}$ means "holds in classical logic" and $\vdash_{\text{IL}}$ similarly "holds in intuitionistic logic." So Heyting algebras are to IL what Boolean algebras are to CL. And also IL is a generalization of CL.

For modal logics the case is not yet so clear. The problem was solved by Kripke, and is now known as *Kripke semantics*. By a *frame* we mean a pair $F = (W, R)$ where W is a set (the set of possible "worlds," the world at different "instants," or different stages of "knowledge") and R a relation on $W, R \subseteq W \times W$ (the "accessibility"). A *model* is then another pair $M = (F, v)$, where F is a frame and where v is a valuation, i.e., a map $v: \mathscr{L} \to W$. Statements which does not include any of the two modal operators $\Box$, $\Diamond$ are denoted by small Roman letters, and modal statements by lowercase Greek ones. The metamathematical statement $M \vDash_w \phi, w \in W$ ("ϕ holds at world w in model M") is then defined by the rules:

1. $M \vDash_w p$ if and only if $w \in v(p)$.
2. $M \vDash_w \neg\phi$ if and only if $M \vDash_w \phi$ does *not* hold.
3. $M \vDash_w (\phi \wedge \psi)$ if and only if $M \vDash_w \phi$ and $M \vDash_w \psi$ both hold.
4. $M \vDash_w \Box\phi$ if and only if

$$\forall w' \in W, \qquad w'Rw: \quad M \vDash_{w'} \phi \tag{41}$$

The restrictions on the modal operators are reflected in similar restrictions on the accessibility relation R as given by

(K) and R

(T) R reflexive

(S4) R reflexive and transitive

(B) R symmetric

(S5) R an equivalence relation

To get a complete description, we must add a rule of inference, to complement the classical *modus ponens* (MP) rule:

$$\mathscr{A}, \quad (\mathscr{A} \Rightarrow \mathscr{B}) \vDash \mathscr{B}$$

this rule is called the rule of *necessitation* (N) and is

$$\text{if} \quad \vDash \phi \quad \text{then} \quad \vDash \Box\phi$$

We will now study the physical implications of (i) the Kripke semantics and (ii) the various modal axioms. After that we will return to an equivalent formulation called *Beth models.*

Let us first look at some possible worlds. A set W very often used in computer science is the knowledge at various times, in which case R is the simple ordering of times, $tRt' \Leftrightarrow t \leq t'$. Typically the system is a computer running some program and the analyst has some incomplete knowledge about the result of the program (will it terminate?, etc.); by making various tests (e.g., running some small programs) the analyst can then increase his information. An application of this is in AI research (artificial intelligence). In physics we could take as "worlds" some regions of space-time, and the accessibility relation would then tell us which regions can receive information from which other regions, i.e., which regions intersects the past light-cone of some observer and which do not. Or, in a somewhat similar setting, but more relevant for quantum cosmology, the worlds could be the spatial 3-manifold at a given "instant"; R then determines which manifolds can evolve into which other manifolds. Many other possibilities exist, but we will restrict ourselves to these examples for the present.

The accessibility relation R tells which worlds we can access from some given world, i.e., a kind of causality. The axioms N and K are unavoidable. Reflexivity of R means $\forall w \in W: wRw$, that is, a world can always be accessed from itself—this is a very reasonable requirement, so we must also demand the axiom T. Transitivity, $\forall w_1, w_2, w_3 \in W: (w_1 R w_2) \wedge (w_2 R w_3) \Rightarrow (w_1 R w_3)$, is clearly also very reasonable from a physical point of view. But here we must stop; we cannot in general require symmetry $wRw' \Leftrightarrow w'Rw$, as it is well known that in general space-times a region may be unaccessible for one observer but this does not imply that the region cannot "access" the first one; think, for instance, of space-times with boundaries (Hawking and Ellis, 1973). So *summa summarum* we end up with the axiom scheme S4. This is very important, as it has been shown that *intuitionistic logic is equivalent to a modal extension of CL satisfying the axiom scheme S*4 (Kleene, 1980). From this we infer the necessity of a description in terms of IL and not CL. Again we see that causality is closely related to IL, and hence Heyting algebras.

Let us round off this discussion with some comments on an equivalent (and very similar) interpretation of modal logics; the so-called *Beth models.*

Let P be a poset, and let $p \in P$; by a *path* through p we mean a subset $A \subseteq P$ such that:

1. $p \in A$.
2. A is *linearly ordered,* i.e., $\forall q, r \in A: (q \preceq r) \vee (r \preceq q)$.
3. A cannot be extended to a larger linearly ordered set.

When we take P to represent the set of *all possible states of knowledge* a path is *complete history of research.* The physical relevance for this is obvious. By a *bar* for p we mean a set $B \subseteq P$ such that all paths through p intersect B, i.e., a bar represents some "unavoidable knowledge," something we cannot help but being able to deduce, no matter which path of research we follow. In symbols, $B = \{q \in P | p \vdash \Box q\}$. Let v be a valuation; then we define a Beth model $\mathscr{M}$ by:

1. $\mathscr{M} \models_p \phi$ if and only if a bar B for p exists such that $B \subseteq v(\phi)$.
2. $\mathscr{M} \models_p (\phi \vee \psi)$ if and only if there exists a bar B for p such that for all $q \in B$ either $\mathscr{M} \models_p \phi$ or $\mathscr{M} \models_p \psi$.

Notice that especially in this formulation the information-theoretic basis of IL is clear. It is this basis that suggests a use in physics. We will make a few comments on this here. Suppose we wanted to make a truly fundamental theory, one from which we could derive all the known laws of nature. Since these laws include those of quantum mechanics, we see that we have to avoid inconsistencies with quantum theory. The fundamental property of quantum mechanics is the notion of indeterminism, in the strong sense that certain values are indeterminate not as a result of our lack of knowledge, but as a result of a fundamental characteristic of nature. Such indeterminism can be introduced in two ways: (1) via stochasticity or (2) via modality. As statistics in the final analysis provides us with a map $Q \to I = [0, 1]$ of the physical parameters in question into the unit interval, we can reexpress this in terms of modalities, as I is a Heyting algebra. Consistency then intervenes again, this time in the guise of causality: If certain statements have indeterminate truth values, how do we prevent situations in which one observer has fixed the value as true and the other as false? The answer comes from Kripke semantics, or more explicitly, with the accessibility relation R. We saw that physical notions of causality in a very abstract sense required this to satisfy the S4 axiom scheme. But then we automatically end up with intuitionistic logic. In this sense *IL is the logical language behind all possible physically fundamental descriptions of nature.*

Example. Local Truth. We give an example of how to introduce modalities. Consider a poset $(P, \preceq)$; we introduce a new partial order $p \sqsubseteq q$ which we read as "p is close to q." Define

$$\mu(p) = \{q | p \sqsubseteq q\} \tag{42}$$

If we have a model $M = (P, v)$ based on P, we can introduce a new connective $\Diamond$ which is interpreted semantically as

$$M \models_p \Diamond\alpha \quad \text{if and only if} \quad \mu(p) \subseteq M(\alpha) \equiv \{q | M \models_p \alpha\} \tag{43}$$

We then have a modal logic provided:

- The sets $\mu(p)$ are contained within the corresponding Alexandrov sets, $\mu(p) \subseteq A^+(p)$.
- The partial order $\sqsubseteq$ is *dense*, i.e.,

$$\forall p, q: \quad p \sqsubseteq q \;\Rightarrow\; \exists r: \quad p \sqsubseteq r \sqsubseteq q$$

- μ "antiharmonizes" with $\preceq$ in the sense that

$$p \preceq q \;\Rightarrow\; \mu(q) \subseteq \mu(p)$$

A concrete example could be a topological space in which p close to q means q is a limit point of $\{p\}$. We could, for instance, define the modality in terms of the p-neighborhoods N_p:

$$M \vDash_p \Diamond\alpha \quad \text{if and only if} \quad \exists N_p: \quad N_p \subseteq M(\alpha) \tag{44}$$

where N_p denotes a p-neighborhood. This construction works in a larger class than just the class of topological spaces, namely the class of *neighborhood spaces*, and even beyond that to arbitrary objects of *categorial topology*.[2] Notice that the Alexandrov sets themselves define a modality if the space is "dense," i.e., if for all x, y a z exists such that $x \preceq z \preceq y$ whenever $x \preceq y$. This once again suggests that causality should be treated as defining a modality [see in this respect also Woodhouse (1973), Bombelli *et al.* (1987), Bombelli and Meyer (1989), Borchers and Sen (1990), and Brightwell and Gregory (1991)], once again hinting at a close relationship between causality and IL.

4.3. Quantum Logic is a Special Case

This section contains the essential theorem of this paper. We have seen that, while the logic underlying our model for quantum gravity was nonclassical, it was also not a quantum logic, but we will now show that it is in fact an extension of quantum logics. First of all we must know how to translate classical formulas into intuitionistic ones. Then we transform the basic requirement and show that it holds in a Heyting algebra.

As mentioned, CL is contained within IL; hence a way to map CL statements into IL ones must exist. In fact several exist, but we will only define one; further examples can be found in Kleene (1980). A well-formed statement is a *prime formula* when it contains no logical symbols whatsoever, i.e., not $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$, $\forall$, $\exists$, $\Box$, $\Diamond$, $\neg$. Clearly such statements are

[2] Categorial topology treats various structures appearing in functional analysis and general topology such as topological spaces, neighborhood spaces, uniformities, simplicial complices, etc., on an equal footing (Preuss, 1988; Kelley, 1975).

trivial to transform from CL to IL. Given an arbitrary well-formed statement $\mathscr{A}$, we define its *prime part* $\mathscr{A}^\circ$ by the following set of inductive rules:

1. If $\mathscr{A}$ is a prime formula, then $\mathscr{A}^\circ = \mathscr{A}$.
2. $(\mathscr{A} \Rightarrow \mathscr{B})^\circ = (\mathscr{A}^\circ \Rightarrow \mathscr{B}^\circ)$.
3. $(\mathscr{A} \wedge \mathscr{B})^\circ = (\mathscr{A}^\circ \wedge \mathscr{B}^\circ)$.
4. $(\neg\mathscr{A})^\circ = \neg(\mathscr{A}^\circ) = \neg\mathscr{A}^\circ$.
5. $(\mathscr{A} \vee \mathscr{B})^\circ = \neg(\neg\mathscr{A}^\circ \wedge \neg\mathscr{B}^\circ)$.
6. If x is a variable and $\mathscr{A}(x)$ is a well-formed statement, then $(\forall x\, \mathscr{A}(x))^\circ = \forall x\, \mathscr{A}^\circ(x)$.
7. Similarly, $(\exists x\, \mathscr{A}(x))^\circ = \neg\forall x\, \neg\mathscr{A}^\circ(x)$.

The requirements 5 and 7 are needed because of the peculiarities of IL. It has been proven by Gentzer and Bernays that

$$\text{if} \quad \vdash_{\text{CL}} \mathscr{A} \quad \text{then} \quad \vdash_{\text{IL}} \mathscr{A}^\circ \tag{45}$$

One very useful way of thinking of this transformation rule is as defining new logical connectives $\vee^\circ$, $\wedge^\circ, \ldots$. Intuitionistic theorems are then obtained from classical ones by the transformation $\vee \to \vee^\circ$, etc. It is this we will use.

Quantum logics had to satisfy the orthomodularity demand

$$a \leq b \;\Rightarrow\; b = a \vee (b \wedge \neg a)$$

but this is written in terms of CL connectives! This can be seen by noting that

$$(a \;\Rightarrow\; b) \vdash_{\text{CL}} (b \;\Leftrightarrow\; (a \vee (b \wedge \neg a)) \tag{46}$$

Or by noting that the following hold in any quantum logic:

$$\vdash a \vee \neg a$$

$$\vdash a \;\Rightarrow\; a \vee b$$

$$\vdash a \wedge b \;\Rightarrow\; a$$

$$\vdash \neg\neg a \;\Leftrightarrow\; a$$

$$\vdash \neg(a \wedge b) \;\Leftrightarrow\; (\neg a) \vee (\neg b)$$

$$\vdash (a \wedge b) \vee (a \wedge c) \;\Rightarrow\; a \wedge (b \vee c)$$

and from the fact that the set of logical connectives is redundant; from $\wedge$ and $\neg$ we can define $a \vee b$ as $\neg((a) \wedge (\neg b))$ and $a \;\Rightarrow\; b$ by $(\neg b) \vee c$. These are just the Gentzer schemes for classical logic. Furthermore, in intuitionistic logic, the set of connectives is not redundant.

In any relative pseudocomplemented lattice we have $a \leq b$ if and only if $a \Rightarrow b$ is true. But many equivalent formulations of this exist, e.g., $(a \Rightarrow b)$ is identical to $(\neg b \Rightarrow \neg a)$, as we know from elementary (classical) mathematics. Transforming the remainder of the requirement into IL terms, we get

$$\neg(\neg a \wedge \neg(b \wedge \neg a))$$

We can now prove the following theorem:

Theorem 1. The following statements are valid:

$$\text{if} \quad a \leq b \quad \text{then} \quad a \vee (b \wedge \neg a) \leq b$$

$$\text{if} \quad \neg b \leq \neg a \quad \text{then} \quad \neg\neg b \geq \neg\neg a \vee (\neg\neg b \wedge \neg a) \leq \neg\neg(a \vee (b \wedge \neg a))$$

for all elements a, b in any Heyting algebra.

Proof. We have

$$\begin{aligned} a \vee (b \wedge \neg a) &= (a \vee b) \wedge (a \vee \neg a) \\ &\leq (a \vee b) \wedge 1 \\ &= a \vee b \end{aligned}$$

Now, if $a \leq b$, then $a \vee b = b$ and we get the result. Similarly, from $\neg b \leq \neg a$ we get

$$\begin{cases} \neg b \vee \neg a = \neg b \\ \neg b \wedge \neg a = \neg a \end{cases}$$

so

$$\begin{cases} \neg(a \wedge b) \geq & \neg b \vee \neg a = \neg b \\ \neg(a \vee b) = & \neg a \end{cases}$$

and the double negations become

$$\begin{cases} \neg\neg b = \neg(\neg b \vee \neg a) = \neg\neg b \wedge \neg\neg a \\ \neg\neg a = \neg\neg(a \vee b) = \neg(\neg a \wedge \neg b) \\ \qquad \leq \neg\neg a \wedge \neg\neg b \end{cases}$$

hence $\neg\neg b \geq \neg\neg a$, but then $\neg\neg a \vee \neg\neg b = \neg\neg b$, whence we derive

$$\begin{aligned} (\neg\neg a \vee \neg a) \wedge (\neg\neg a \vee \neg\neg b) &= (\neg\neg a \vee \neg a) \wedge \neg b \\ &\leq \neg\neg b \end{aligned}$$

which was the desired result.

Note that when we impose classical connectives $\neg\neg a = a$ we get $\neg(a \leq b) \Rightarrow (\neg b \leq \neg a)$ and we obtain the usual orthomodularity demand. Hence in any Heyting algebra we obtain a generalization of the orthomodularity requirement, which, moreover, reduces to the usual one in the so-called *stable sector* of intuitionistic logic, namely when $\neg\neg a = a$. Also note that the inequalities only hold when $a \leq b$, as this was used in the proof, so contrary to the classical case, we do not have $b = a \vee (b \wedge \neg a)$ for all pairs a, b, hence the distributivity *appears* to have become weakened, which, of course, it has not; the apparent weakening results from the use of nonclassical connectives!

It is perhaps not very surprising that Heyting algebras contain quantum logics as special cases, as it has been shown (Binder and Pták, 1990) that any quantum logic can be represented by the collection of clopen sets (i.e., sets which are simultaneously closed and open) in some closure space, and we have seen that the open sets themselves form a Heyting algebra. Since the collection of clopen sets forms a subset of the set of open sets, the result follows.

4.4. Connection with Quantum Nets

Recently, Finkelstein has proposed as fundamental structure the concept of a *quantum net* (Finkelstein and Finkelstein, 1983; Finkelstein, 1987, 1989; Finkelstein and Hallidy, 1991). An essential ingredient in this model is the so-called *interactive logic*, a logic which includes Boolean logic as well as projective geometry (and hence quantum logic). The model is based on the theory of automata. States are denoted by $S, S', S'', \ldots$, and *controls* $C, C', C'', \ldots$, which are arrows between the states; we write $\langle S|C|S'\rangle = 1$ if C is an arrow going from S to S' and $\langle S|C|S'\rangle = 0$ if C does not connect S and S'; in other words, C is the "transition matrix."[3] Clearly we can represent C by its "matrix elements" $\langle S|C|S'\rangle$ as a Boolean matrix, i.e., a matrix taking its values in a Boolean algebra. The states $S, S', S'', \ldots$ are sets representing the knowledge at a certain stage. The *Boolean product* $C \cdot C'$ of C, C' is defined as

$$\langle S|(C \cdot C')|S'\rangle = \bigvee_{S''} \langle S|C|S''\rangle\langle S''|C|S'\rangle \tag{47}$$

and we define the space Q of *control sequences* as

$$Q = \{1, C, C \cdot C, C \cdot C \cdot C, \ldots\} \tag{48}$$

[3]Finkelstein uses the notation $S:C:S'$ for $<S|C|S'\rangle$, but I have decided to opt for the more "physical-looking" formulation.

Control sequences can be multiplied in a natural way, by noticing that Q can be written $Q = 1 \vee C \vee C \cdot C \vee \cdots$, where $\vee$ is the set-theoretic join. Write $\Diamond Q \rightarrow Q'$ if $QQ' \neq 0$[4]; we could also write this in a more metamathematical way as $Q \vdash \Diamond Q'$, i.e., given Q, Q' is a possible future control sequence of the system. Write $Q \perp Q'$ if $QQ' = 0$ (Finkelstein's o-relation); this relation is in general not symmetric.

Also define the *Kleene algebra* K as

$$K = \{f: Q \rightarrow \mathbf{2}\} = \mathbf{2}^Q = \mathscr{P}(Q) \tag{49}$$

where $\mathbf{2}$ denotes the set $\{0, 1\}$. Members of K are predicates about the control sequences, and it is from here we get the interactive predicates. Equivalently we could say that K is a class of Q's and that the control sequences are elements of K; this reflects a general structure: the points in a space can be considered as "subjects" and the sets as "predicates"; more on this later. The Kleene algebra inherits a semigroup structure from Q.

If Q belongs to K, then we can ask for the class K' of *all possible futures* Q' of Q, i.e., a Beth bar $\{Q' | Q \vdash \Diamond Q'\}$, and we can ask for the class $K^{\perp}$ consisting of all *excluded futures*, i.e.,

$$K^{\perp} = \{Q' | Q \vdash \neg \Diamond Q'\} \tag{50}$$

(here Finkelstein uses the notation $K \vdash$). Define the relation

$$K \sim K' \quad \text{iff} \quad K^{\perp} = K'^{\perp} \tag{51}$$

We can now get a Heyting algebra structure as follows. We must distinguish between initial and final classes: in $Q \perp Q'$, Q is initial while Q' is final. Let A denote a class of initial states and X one of final states, such that $A \perp X$ holds. Consider subclasses a, x of A, X, respectively, and define

$$a^{\perp} = \{X' \subseteq X | \forall A' \in a: A' \perp X'\} \tag{52}$$

$$^{\perp}x = \{A' \subseteq A | \forall X' \in x: A' \perp X'\} \tag{53}$$

This is essentially the same construction as the *polar* and *prepolar* of a subspace of a Banach space (or indeed any module) X:

$$A^{\circ} = \{f \in X^* | \forall a \in A: f(a) = 0\}$$

$$^{\circ}B = \{a \in X | \forall f \in B: f(x) = 0\}$$

where $A \subseteq X$, $B \subseteq X^*$. I will return to this point in a moment. The classes $a^{\perp}$ and $^{\perp}x$ are related to the possibility operator as follows:

$$a^{\perp} = \{x | \neg(a \vdash \Diamond x)\} \tag{54}$$

$$^{\perp}x = \{a | \neg(a \vdash \Diamond x)\} \tag{55}$$

[4]Finkelstein writes $Q' : p : Q$, which could then also be rendered $\langle Q | \Diamond | Q' \rangle$ in our notation.

i.e., $a^{\perp}$ contains the impossible futures, while ${}^{\perp}x$ contains the impossible pasts. One easily shows that $\perp$ reverses inclusion:

$$a \leq b \;\Rightarrow\; b^{\perp} \leq a^{\perp} \tag{56}$$

$$x \leq y \;\Rightarrow\; {}^{\perp}y \leq {}^{\perp}x \tag{57}$$

from which it follows that

$${}^{\perp}a^{\perp\perp} = a^{\perp} \tag{58}$$

$${}^{\perp\perp}x^{\perp} = {}^{\perp}x \tag{59}$$

These are just the usual Heyting algebra rules. Finkelstein goes on to introduce a closure operator

$${}^{-}a = {}^{\perp}(a^{\perp}) \tag{60}$$

$$x^{-} = ({}^{\perp}x)^{\perp}) \tag{61}$$

This is just the closure belonging to the double negation modality. We then get a lattice of *initially closed* classes, i.e., classes for which ${}^{-}a = a$, by defining

$$a \vee b \equiv{}^{-} (a \cup b) \tag{62}$$

which is essentially just the Heyting algebra definition for a topological space (we work with closed sets so we take closure instead of interior). One can then prove

$$a^{\perp} = b^{\perp} \;\Leftrightarrow\; {}^{-}a = {}^{-}b \tag{63}$$

Similarly we have a lattice of *finally closed* classes, i.e., classes such that $x^{-} = x$. We can now see that Finkelstein's quantum net formalism fits very naturally in with our own ideas. Let us note the fact that Banach spaces exhibit similar structures [with pseudocomplement defined by polar or prepolar; see, for instance, Rudin (1973)]; this suggests a close relationship with algebraic quantum mechanics, where we use not Hilbert spaces, but operator algebras of observables (these algebras are also Banach spaces). More will be said regarding this point later. As a final comment, let us note that the appearance of Heyting algebras does not come so much from the modal operator $\Diamond$, but rather from the relation $\perp$, hence whenever we have a relation like this we should expect intuitionistic logic to be the natural language! But such a notion of inaccessible pasts or futures is inherent in the concept of causality, hence from causality alone we should expect IL to be that natural logical structure. The appropriate definitions would be as follows. Denote the future light cone of a point p by $C^{+}(p)$ and define

$$p \perp q \qquad \text{iff} \quad q \notin C^{+}(p) \tag{64}$$

Note that this relation is in general not symmetric. Following in Finkelstein's footsteps, we introduce

$$
\begin{aligned}
p^{\perp} &= \{q \mid p \perp q\} = \{q \mid q \notin C^{+}(p)\} = M \backslash C^{+}(p) \\
{}^{\perp}q &= \{p \mid p \perp q\} = \{p \mid q \notin C^{+}(p)\}
\end{aligned}
$$

where M denotes space-time. Hence $p^{\perp}$ is the set of impossible future positions and ${}^{\perp}q$ is the set of impossible past positions. This notion extends to arbitrary regions of M. Define the causal partial order on these regions by

$$
A \preceq B \qquad \text{iff} \quad B \subseteq C^{+}(A) = \bigcup_{p \in A} C^{+}(p)
$$

We have

$$
\begin{aligned}
A^{\perp} &= \{q \mid \forall p \in Q : p \perp q\} \\
{}^{\perp}A &= \{p \mid \forall q \in A : p \perp q\}
\end{aligned}
$$

As Finkelstein points out, the fact that $\perp$ reverses inclusions is completely independent of the relation; similarly, as the triple-$\perp$ identities follow from these inequalities, we see that the Heyting nature of $\perp$ is quite general and hence also holds in our case.

Finkelstein also writes down an algorithm for obtaining the logical language (what he calls interactive logic, but which we have seen is just a special case of intuitionistic logic) of an automaton. This is very useful as, e.g., 't Hooft (1990) recently proposed a model for quantum gravity based on a cellular automaton. The algorithm is of a similar nature to that given by Vickers (1989) for obtaining Heyting algebras from "subbasic operations" and relations.

5. TOPOI AND QUANTUM LOGIC

We will here briefly investigate further the mathematical structure of quantum logics. The essential element in the definition of quantum logics is the definition of the map $m: Q \to I$, where Q is the set of questions and I is the unit interval. Now, the set of questions has the form $A \times \mathscr{P}(A)$ for some set A (usually the set of reals). Given any map $f: A \times B \to I$, where B is any set, we can construct a unique map $A \times B \to A \times \mathscr{P}(A)$, denoted by $1_A \times \hat{f}$. The diagram is

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\;f\;} & I \\
{\scriptstyle 1_A \times \hat{f}} \downarrow & \nearrow_{m} & \\
A \times \mathscr{P}(A) & &
\end{array}
$$

The map $\hat{f}$ is given by

$$\hat{f}(b) = \{a \in A \mid f(a, b) = 1\} \tag{65}$$

and is known as the *exponential transpose* of f. The general setup is thus

$$\begin{array}{ccc} A \times B & \xrightarrow{f} & \Omega \\ {\scriptstyle 1_A \times \hat{f}} \downarrow & \nearrow {\scriptstyle \mathrm{ev}_A} & \\ A \times \mathscr{P}(A) & & \end{array}$$

where Ω is some set of truth values, and ev_A is given, in the example above, by $\mathrm{ev}_A(a, X) = 1$ if and only if $a \in X$, i.e., ev_A evaluates the truth of the statement $a \in X$, $X \subseteq A$, whereas in the theory of sets we take $\Omega = \mathbf{2} = \{0, 1\}$ and the maps $A \to \Omega$ are just characteristic functions χ_B for B's subsets of A. Such a structure is very important in modern mathematics, and it is related to the concept of a *topos* (plural: topoi) (Goldblatt, 1984; Bell, 1988; Chapman and Rowbotton, 1992; Vickers, 1989). A topos is defined as a category with the following properties:

1. It has *binary products*, i.e., $A \times B$ is defined for all objects A, B.
2. It has a *terminal object* 1, i.e., for each object A there is a unique arrow $A \to 1$.
3. It has what is known as a *subobject classifier* Ω.
4. It has *power objects*, i.e., $(\mathscr{P}(A), \mathrm{ev}_A)$ is defined for all objects A.

We will now give the mathematical definition of these terms, and then we will see their physical interpretation.

Since we will only be dealing with objects which are again sets, the binary product is just the Cartesian product and is thus well defined for all sets A, B. As an example of binary product in another category than that of sets, we can consider a poset. This can be turned into a category very simply, by drawing an arrow $p \to q$ if and only if $p \leq q$. The binary product is then simply $p \wedge q$.

A terminal object in a category of sets is just a singleton set, i.e., a set of the form $\{x\}$. In a poset it would be a maximal element.

The concept of a subobject classifier is a bit more complicated. In standard set theory $\Omega = \mathbf{2} \equiv \{0, 1\}$. It can be used to define *subobjects* by means of a "characteristic function." The set Ω represents the set of truth values, and a subobject is then characterized by an arrow $\chi_B : A \to \Omega$; $\chi_B(a)$ is true if and only if $a \in B$. A more rigorous definition can be found in Goldblatt (1984), Bell (1988), Chapman and Rowbotton (1992), and Vickers (1989). In a category of sets, a subobject is just a subset.

Power objects are defined as the collection of all subobjects together with an evaluation map $\mathrm{ev}_A : \mathscr{P}(A) \to \Omega$; the collection $\mathscr{P}(A)$ is isomorphic to Ω^A, which is the generalization of the well-known statements for elementary set theory $\mathscr{P}(A) \simeq \mathbf{2}^A$. Let us now consider the physical relevance of these purely mathematical axioms. Binary products allow us to perform consecutive measurements; they are conjunctions. Terminal objects are to be interpreted as elements of maximal information, in a sense to be made precise later. The subobject classifier is the set of truth values and is thus taken to be $\Omega = I = [0, 1]$ in general. This leads to a slight problem. The class $\mathscr{P}(A)$ then becomes isomorphic to, not the set-theoretic powerset, but the functor category Ω^A. Luckily, a very simple and elegant interpretation of this exists. An element of the set I^X for some set X is known as a *fuzzy set*. Our conclusion must be: *we must use a topos with* $\Omega = I$; *the subobjects are then fuzzy sets and the logic is non-Boolean.* Note also that I is a Heyting algebra. In general, the logic behind a topos is intuitionistic, i.e., based on Heyting algebras, and only in special cases is it Boolean. The fundamental structure for a logical analysis of quantum mechanics is not really an orthomodular lattice, but rather a topos. In any case we end up with intuitionistic logic as the true underlying logical system. The use of a probability measure m can be reexpressed as the use of fuzzy sets, which are just the natural subobjects in this topos. The lack of distributivity in quantum logics is more due to our insistence on working with classical connectives: in the "projection" of IL unto CL we lose some information, which we interpret as a loss of distributivity. The theory of fuzzy sets can be found in Šostak (1990) and the references therein. I think that this formulation is also very well suited for a discussion of field theory in terms of logic; work is in progress on this subject.

The prime example of a classical logic is the powerset $\mathscr{P}(A)$ of some set A; this is the most natural class of sets connected with a given set, and one could thus argue that Boolean logic is the most general kind of logic, as we have seen that we had to consider restricted classes of sets (topologies, etc.) in order to get a Heyting algebra. But this is wrong! A topos generalizes the notion of a set, and here the "powersets" $\mathscr{P}A \simeq \Omega^A$ are Heyting algebras, and not in general Boolean. So, if we restrict ourselves to sets, the most natural logic is the Boolean one, unless we include some kind of structure (the canonical one being a topology), but if we are willing to accept the notion of a generalized set, then we automatically end up with Heyting algebras. This is shown in Table I.

Table I. Hierarchy of Structures and Logics

Ω	Structure	Logic
$\mathbf{2} \equiv \{0, 1\}$	Sets	Boolean logic
$I \equiv [0, 1]$	Fuzzy sets	Fuzzy logic
Arbitrary	Generalized sets	Intuitionistic logic

6. ON YANG–MILLS FIELDS

As is well known (Choquet-Bruhat *et al.*, 1982; Göckeler and Schücker, 1989; Warner, 1983), Yang–Mills theory can be formulated in a very beautiful geometrical way by the use of fiber bundles. It has also been shown that this setting can be generalized to quantum groups, but in this case the fibers do not have to be isomorphic (Müller, 1992). Finally it should be mentioned that ideas along these lines have been proposed in the random dynamics project in the guise of the so-called "gauge-glasses," which basically consist of a collection of regions in which the fibers are isomorphic, but these regions and the structure of the fibers are random, and hence one region of space would have an $SU(2)$ bundle structure while a neighboring one could have an $SU(3)$ (Nielsen and Brene, n.d.; Froggat and Nielsen, 1991). Here I want to argue that the frame proposed in this paper can deal with these possibilities in a very natural way.

The essential concepts are a collection of sets $\{U_\alpha\}$ and a map $F: U_\alpha \mapsto \hat{U}_\alpha$ such that

$$U_\alpha \hookrightarrow U_\beta \quad \text{gets mapped into} \quad \hat{U}_\beta \to \hat{U}_\alpha \tag{66}$$

The map $\hat{U}_\beta \to \hat{U}_\alpha$ will be denoted by F^α_β. We furthermore require that, when $U_\alpha \hookrightarrow U_\beta \hookrightarrow U_\gamma$ is given, then

$$F^\alpha_\gamma = F^\alpha_\beta \circ F^\beta_\gamma \tag{67}$$

this can be reformulated as stating that the diagram

$$\begin{array}{ccc} \hat{U}_\gamma & \xrightarrow{F^\beta_\gamma} & \hat{U}_\beta \\ & {\scriptstyle F^\alpha_\beta}\searrow \quad \swarrow{\scriptstyle F^\alpha_\beta} & \\ & \hat{U}_\alpha & \end{array}$$

commutes.[5] The idea is that $U_\alpha \simeq \hat{U}_\alpha \times \mathscr{G}_\alpha$, where $\mathscr{G}_\alpha$ is some Lie group (or quantum group), i.e., the space $B = \bigcup_\alpha U_\alpha$ looks locally like a trivial bundle (a Cartesian product). We have a natural projection $\pi_\alpha: U_\alpha = \hat{U}_\alpha \times \mathscr{G}_\alpha \to \hat{U}_\alpha$, and these projections just make up the map F above. Whenever the collection $\{U_\alpha\}$ consists of points only, $U_\alpha = \{x_\alpha\}$, we have a fiber bundle with fiber $\mathscr{G}_x$ at each point. The set $\mathscr{G}_\alpha$ is known as the *stalk*.

[5]Categorially speaking, a presheaf is a contravariant functor into a category of sets. Any poset is automatically also a category, so *a fortiori* space-time is a category, with the partial ordering given by causality, or with the structure imposed on it by the topology. Considering space-time solely as a topological space, we could let M be the corresponding *frame* of open sets or some similar structure (Vickers, 1989).

The structure we have defined here is known as a *presheaf*. The set $M = \bigcup_\alpha \hat{U}_\alpha$ will be taken to be the space-time manifold. We will now see how this fits in to the frame defined by our model, i.e., we will show that a presheaf is a topos. Consider for each α the set

$$\Omega(\hat{U}_\alpha) = \{f: \hat{U}_\alpha \to B\} \tag{68}$$

The elements of this are known as *cosieves* on $\hat{U}_\alpha$. This is our candidate for a subobject classifier. The setup we have now is namely this: let f be a function $\hat{U}_\alpha \times \hat{U}_\beta \to \Omega = \{\Omega(\hat{U}_\gamma)\}$; as all the objects are sets, we have a unique function $\hat{f}$ and an evaluation map ev_α such that the diagram

$$\begin{array}{ccc} \hat{U}_\alpha \times \hat{U}_\beta & \xrightarrow{f} & \Omega \\ {\scriptstyle 1_\alpha \times \hat{f}} \searrow & & \nearrow {\scriptstyle \mathrm{ev}_\alpha} \\ & \hat{U}_\alpha \times \mathscr{P}\hat{U}_\beta & \end{array}$$

commutes, where 1_α is the identity function on $\hat{U}_\alpha$ and where

$$\mathscr{P}\hat{U}_\beta \equiv \{\hat{U}_\beta\text{-valued functions}\} \tag{69}$$

This gives us a topos. Hence we can formulate the relevant mathematical structure in terms of topoi, which opens up the possibility of using the very powerful methods developed in recent years by mathematicians; in particular, we can develop a purely logical formulation.

Now, the Yang–Mills field is a connection on a principal bundle, and a connection can be defined "semialgebraically" as a family of subspaces of the tangent bundle satisfying certain requirements (Choquet-Bruhat *et al.*, 1982; Göckeler and Schücker, 1989; Warner, 1983). Similarly, the field strength tensor is nothing but the curvature of this connection. So we must give meaning to a tangent bundle. This can also be done algebraically, namely as follows (Warner, 1983). Define an equivalence relation on the set of germs of functions on $\hat{U}_\alpha$ (this is again a presheaf, by the way), as

$$f \sim g \Leftrightarrow f|_{\hat{U}_\alpha} = g|_{\hat{U}_\alpha} \tag{70}$$

and construct the sets

$$\hat{\mathscr{F}}_\alpha = \mathscr{P}\hat{U}_\alpha / \sim \tag{71}$$

and

$$\mathscr{F}_\alpha = \{f \mid f \sim 0\} \tag{72}$$

It is easily seen that $\mathscr{F}_\alpha$ is an ideal in $\hat{\mathscr{F}}_\alpha$, hence the following constructions are legal

$$T\hat{U}_\alpha = (\mathscr{F}_\alpha / \mathscr{F}_\alpha^2)^* \tag{73}$$

$$T^*\hat{U}_\alpha = (T\hat{U}_\alpha)^* \tag{74}$$

where * denotes the dual module $N^* = \mathrm{Hom}(N, \mathsf{R}/\mathsf{Z})$, where N is some module, and R and Z the reals and the integers, respectively. These sets are the stalks of what we will term the *tangent* and *cotangent presheaves.* Put

$$TM = \bigcup_\alpha T\hat{U}_\alpha \tag{75}$$

$$T^*M = \bigcup_\alpha T^*\hat{U}_\alpha \tag{76}$$

Elements of these presheaves are known as tangent and cotangent elements, respectively. Having defined these, we can go on to define p-forms in the usual way using the p-fold tensor products of T^*M.

Connections can then be defined as families of subspaces $\{H_\alpha\} \equiv \{H(U_\alpha)\}$ of $T\hat{U}_\alpha$ satisfying:

- They are the complements of the spaces V_α defined by $V_\alpha = \{(v_M, v_\mathscr{G}) | v_M = 0\}$, where $v_M \in T\hat{U}_\alpha$ and $v_\mathscr{G} \in T\mathscr{G}_\alpha$, i.e., $H_\alpha \oplus V_\alpha = TU_\alpha$.
- They are locally spanned by C^∞-vector fields on B.
- Under the right action of $\mathscr{G}_\alpha$ they transform as

$$H_{g\alpha} \equiv H(gU_\alpha) = T\tilde{R}_g(H_\alpha) \tag{77}$$

where $\tilde{R}_g$ is the right action of the structure group on the bundle. In a local trivialization (U_r, f_{rx}) this is defined by

$$(\tilde{R}_g p)_r \equiv f_{rx}^{-1}(R_g g_r) = f_{rx}^{-1}(g_r g) \tag{78}$$

with $p \in \pi^{-1}(x)$ an element in the fiber, $x \in U_r$, and

$$g_r \equiv f_{rx}(p)$$

See Göckeler and Schücker (1989).

The *vertical spaces* V_α are the kernels of the projections π_α (or rather the projections $\hat{\pi}$ of the tangent presheaves induced by π). Given such a collection of *horizontal spaces* H_α, a 1-form taking values in $\mathscr{G}_\alpha$ then exists such that the horizontal spaces are the kernels of these, i.e., $H_\alpha = \ker \mathscr{A}_\alpha$ with $\mathscr{A}_\alpha \colon TU_\alpha \to \mathscr{G}_\alpha$. Given a derivation d_α on the Grassmann algebra of forms, we then form the covariant derivative $D_\alpha = d_\alpha + [\mathscr{A}_\alpha, \cdot\,]$; the curvature is then the covariant derivative of the connection 1-form, $F_\alpha = D_\alpha \mathscr{A}_\sigma = d_\alpha \mathscr{A}_\alpha + [\mathscr{A}_\alpha, \mathscr{A}_\alpha]$.

We could then go on to investigate whether the Bianchi identities need modification or not, study "holonomy," etc. Furthermore, since anomalies can be cast in a purely algebraic geometric form, it should be possible to translate these into topos language, which would then yield, in the final analysis, a purely logical formulation of anomalies.

The natural logical concept belonging to these kinds of structures is the notion of "local truth," which can be expressed in physical terms as, "it has been observed locally that . . ."; this actually implies that we are dealing with a higher-order logic. To any topos $\mathscr{E}$ we have associated a logical language, the so-called *internal language* of the topos. This language is *local* in the sense that only logical operations on statements of the same *type* are allowed. To each set U_α corresponds a type $\mathbf{U}_\alpha$ in the internal language $\mathscr{L}(\mathscr{E})$; similarly, to Ω corresponds a type $\mathbf{\Omega}$, to $\top \in \Omega$ corresponds *true*, and we define *power types* $\mathscr{P}\boldsymbol{U}_\alpha$, *product types* $\mathbf{U}_\alpha \times \cdots \times \mathbf{U}_\beta$, and *function types* $\boldsymbol{U}_\alpha \to \boldsymbol{U}_\beta$. I refer to Bell (1988) for further details.

The subobject classifier Ω gives the set of possible truth values, and it can be proven that Ω is a Heyting algebra in any topos $\mathscr{E}$ (see Goldblatt, 1984, or Bell, 1988). Now, a topos need not refer to any topological space at all, but one can prove some interesting results. We say that a topos is *localic* if it is isomorphic to a topos of sheaves over some *locale* H, i.e., $\mathscr{E}$ is localic provided $\mathscr{E} \simeq \mathbf{Sh}(H)$, where $\mathbf{Sh}(H)$ denotes the topos of sheaves over some locale H; note that H need not be a topological space. Similarly we say that Ω is *spatial* if it is complete as a lattice and

$$\forall a, b \in \Omega, a \neq b \exists c \in \Omega: \quad [(a \wedge b \leq c \;\Rightarrow\; a \leq c \vee b \leq c) \wedge (a \leq c \;\Rightarrow\; b \nleq c)] \tag{79}$$

One can then prove (Bell, 1988) that $\mathscr{E} \simeq \mathbf{Sh}(X)$, where X is some *topological space* if and only if $\mathscr{E}$ is localic and Ω is spatial. Another important case is when the topos is isomorphic to the collection of presheaves over some poset. We say that Ω is *Alexandrov* if it is complete and

$$\forall a, b \in \Omega, a \neq b \exists c \in \Omega:$$
$$\left[\left(\forall x_i : \bigwedge_{i \in I} x_i \leq c \;\Rightarrow\; \exists i \in I : x_i \leq c\right) \wedge (a \wedge b \leq c) \wedge (a \leq c \;\Rightarrow\; b \nleq c)\right] \tag{80}$$

One can then similarly prove that $\mathscr{E} \simeq \mathbf{PreSh}(P)$, where $\mathbf{PreSh}(P)$ denotes the collection of presheaves over some poset P, if and only if $\mathscr{E}$ is localic and Ω is Alexandrov. A *locale* is a concept from *pointless topology*; essentially it consists of a collection of open sets, i.e., a topology (or more generally a *frame*, which is defined as a complete Heyting algebra), together with a collection of maps $x: L \to \mathbf{2}$, where L denotes the open sets. The elements x are known as the points. A Heyting algebra which is homeomorphic to a locale is then localic. Similarly, the word "spatial" refers to the way one usually does topology, in which the points are primitive and the opens are defined in terms of these. I refer to Vickers (1989) for further details. Locales are closely connected with algebraic geometry (Rosenthal and Niefield, 1989).

6.1. Algebraic Quantum Mechanics and C^*-Algebras

The general idea behind algebraic quantum field theory is to avoid the Hilbert space by considering families of operator algebras defined abstractly. To each region of space-time U is then associated an algebra $\mathscr{A}(U)$; usually the regions are open sets with compact closure and the algebras are C^*-algebras. The axioms are:

- Isotony: $U \subseteq V \Rightarrow \mathscr{A}(U) \subseteq \mathscr{A}(V)$.
- Local commutativity: if U, V are completely spacelike apart, then $[\mathscr{A}(U), \mathscr{A}(V)] = 0$.
- Completeness: $\mathscr{A} = \bigcup_U \mathscr{A}(U)$ is completed to a C^*-algebra which contains *all* observables.
- Lorentz covariance: the Poincaré group is represented by automorphisms $L: A \mapsto A^L$ such that $\mathscr{A}(U)^L = \mathscr{A}(LU)$.
- Primitivity: $\mathscr{A}$ admits a faithful algebraically irreducible representation, i.e., a faithful representation such that only the trivial ideals $\{0\}$, $\mathscr{A}$ are invariant.

The isotony requirement implies that we have to reverse the order to get a topos. Define $U \geq V$ to mean $U \subseteq V$; then, denoting space-time by M, we have a poset category $(M, \geq) = (M, \subseteq)^{\mathrm{op}}$, and we can thus formulate the axioms as stating we have a contravariant functor $\mathscr{A}(\,\cdot\,)^{-1}: (M, \geq) \to \mathbf{C^*Alg}$, where $\mathbf{C^*Alg}$ denotes the category of C^*-algebras. Since we are only interested in open sets, we can formulate this a presheaf over a locale, $\mathcal{O}(X)$.

Admittedly the definition of $\geq$ is somewhat unaesthetic, but is needed if we only consider a locale; note, however, that physics provides us with one extra, highly important structure, namely *causality*. This establishes a new partial ordering of space-time, and hence of our locale $\mathcal{O}(X)$. Letting $C^+(U)$ denote the forward cone of a set U, we say

$$U \preceq V \qquad \text{iff} \quad V \subseteq C^+(U) \tag{81}$$

Following Haag and Kastler (1964), we demand

$$U \preceq V \Rightarrow \mathscr{A}(V) \subseteq \mathscr{A}(U) \tag{82}$$

The presheaf structure then follows naturally.

The elements of $\mathscr{A}(U)$ are the observables that can be observed in the region U, i.e., locally, and *all* statements are then of a purely local nature. The topos structure is then provided by the following sets

$$\Omega(U) = \{f\colon U \to \mathscr{A}\} \quad \text{(the set of } C^*\text{-algebra-valued functions)} \tag{83}$$

$$\mathscr{P}U = \Omega^U \tag{84}$$

A different formulation can also be given. The Hamiltonian is, in the final analysis, always given by an expression like

$$H \sim \sum_k \left[a_k^\dagger a_k + \sum_l (a_k^\dagger a_{l-k} a_l + a_{l-k}^\dagger a_l^\dagger a_k + \cdots) + \cdots \right] \tag{85}$$

where the sums are over all quantum numbers (if these are continuous, we of course interpret the sum as an integral). Hence at each point in k-space, we have a fiber $\mathscr{A}_k$ consisting of a C^*-algebra, in such a way that

$$[\mathscr{A}_k, \mathscr{A}_l] = 0 \qquad \text{if} \quad k \neq l \tag{86}$$

and when $k = l$ we have some algebraic relations. For instance, if we have at each point just an ordinary harmonic oscillator, we would get

$$[a_k, a_k] = [a_k^\dagger, a_k^\dagger] = 0$$

$$[a_k^\dagger, a_k] = 1$$

while if we also have some gauge symmetry, we could write the elements of $\mathscr{A}_k$ as $a_{k,a}$, where

$$[a_{k,a}, a_{k,b}] = ic_{ab}^c a_{k,c}$$

$$[a_{k,a}^\dagger, a_{k,b}] = \delta_{ab}$$

and similarly we could consider supersymmetric systems, etc. In this way we have a natural fiber formulation of quantum systems, which then gives us a natural topos formulation. The Hamiltonian (or the action) should then be some kind of "characteristic class" of this bundle structure:

$$H \sim J^{kl} \mathscr{A}_k \mathscr{A}_l + \xi^{klm} \mathscr{A}_k \mathscr{A}_l \mathscr{A}_m + \cdots$$

where the summation over repeated indices is understood and

$$J^{kl} \mathscr{A}_k \mathscr{A}_l \equiv \sum_{kl} (\alpha a_k^\dagger a_l + \beta a_k a_l^\dagger + \gamma a_k a_l + \delta a_k^\dagger a_l^\dagger)$$

with a, β, γ, δ some complex functions of k, l; normally we would set $\alpha(k, l) = -\beta(k, l) = \frac{1}{2}\delta_{kl}$ and $\gamma = \delta = 0$. This last formulation has the advantage of only introducing the truly fundamental quantities, namely the second quantization operators $a_k, a_k^\dagger$—any other quantity can be formed from these. From this we can go on applying the usual fiber bundle techniques (see Choquet-Bruhat *et al.*, 1982, or Göckeler and Schücker, 1989); the "gauge transformations" would now be general coordinate transformations, but this time in a noncommutative geometry (we can consider the operators $a_k^\dagger$, a_k as coordinates in this space).

This research has only just begun and naturally much needs to be done; this paper just outlines the mathematical frame in which to work.

7. CONCLUSION

We have seen that the study of quantum gravity necessitates a need for a more convenient description of the measurement process, differing from the usual Copenhagen interpretation. This led us to investigate the foundations of quantum logics. We showed that this could be considered as a special case of what is known as intuitionistic logic, which again can be seen as a modal extension of classical logic. Thus we ended up with a formulation of quantum mechanics in terms of intuitionistic logic/modal logic and we finished by further making explicit the mathematical frame in which we have to work. This turned out to be the theory of topoi and of fuzzy sets. We then went on to show how Yang–Mills theory, algebraic quantum mechanics, and even quantum groups fitted in. Thus we seem to have established a common language for all kinds of quantum theory, which should be very well suited for handling the problems arising when quantizing gravity. This language is closely connected with causality. It should be noted that this could only be achieved by abandoning classical mathematics as well as classical physics: instead of sets we have topoi, which can be seen as generalized sets (categorical set theory, local set theory, etc.), implying the use of intuitionistic logic which generalizes classical logic. Further research along these lines needs to be done.

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